The density wave in a car-following model

2002 J. Phys. A: Math. Gen. 35 4495

(http://iopscience.iop.org/0305-4470/35/20/309)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 4495–4500 PII: S0305-4470(02)32215-7

The density wave in a car-following model

Xianjun Zhou, Zhongzhu Liu and Jun Luo

Department of Physics, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China

Received 19 December 2001, in final form 5 April 2002
Published 10 May 2002
Online at stacks.iop.org/JPhysA/35/4495

Abstract

According to the optimal-velocity model, the condition for stable traffic flow is deduced. Nonlinear analysis shows that the density fluctuation in traffic flow itself induces two types of local density waves. A weak fluctuation occurring near the stability state in a wide range of headways forms a soliton determined by the Korteweg–de Vries (KdV) equation. The appearance of such a soliton shows that drivers tend to reach the safety distance when they are away from it. This density wave degenerates to a travelling wave at the critical point. A strong fluctuation occurring around the critical point forms a kink or a soliton determined by the modified Korteweg–de Vries (MKdV) equation.

PACS numbers: 45.70.Vn, 02.30.Ik, 02.30.Jr, 05.45.Yv

1. Introduction

Traffic problems have attracted the interest of a rapidly growing community of physicists over many years [1–14, 17, 23–25]. Traffic flow, a many-body system of strongly interacting vehicles, shows various complex behaviours. Many physical phenomena, such as nonlinear waves and non-equilibrium phase transitions, have been revealed by recent studies. At present, traffic problems have been investigated by many models: the car-following models [1–3], the cellular automaton models [4–6], the gas kinetic models [7, 8], the hydrodynamic models [9–12], and so on. Among these models, the car-following model has often produced some important results owing to its concise form.

Vehicle density can fluctuate for various reasons to form a density wave in a traffic flow, which connects with traffic jams. Lately, Kurtze and Hong [11] have derived the Korteweg–de Vries (KdV) equation from the hydrodynamic model by nonlinear analysis. They concluded that the density wave has a soliton form. Komatsu and Sasa [13] have obtained the modified Korteweg–de Vries (MKdV) equation from the car-following model. They showed that the density wave has a kink shape. Also, Muramatsu and Nagatani [14] derived the KdV and MKdV equations through investigating the traffic flow under the open boundary condition.

The optimal-velocity (OV) model was proposed to describe a traffic flow more closely on the basis of the car-following model by writing the headway as a function of high-order derivatives of vehicle density. The nonlinear theory shows that the soliton will be present in a system with nonlinearity and dispersion when both factors balance each other [15, 16]. The OV model itself is a nonlinear one, and the high-order derivatives of vehicle density provide the dispersive term. Thus, the soliton can appear in the car-following model.

In the OV model, the OV function increases monotonically to its maximal value and has a turning point (i.e. critical point) which corresponds to the safety distance. In this paper, we will discuss the OV model in detail. The nonlinear analysis shows that a weak fluctuation occurring in a wide range of headways forms a soliton determined by the KdV equation near the stability state. It will degenerate to a travelling wave at the critical point. On the other hand, a strong fluctuation occurring around the critical point will form a kink or soliton wave determined by the MKdV equation. It is evident that a soliton with the up wave crest indicates the local density will increase at a position with a larger vehicle distance than the safety distance. The result shows that drivers tend to reduce the distance if there is enough space in front of them. In contrast, drivers will increase the vehicle distance at the position where a soliton with the down wave crest appears. Subsequently, spectrum parameter evolution has been followed by the direct approach, which shows that the amplitude of the soliton tends to increase. In addition, the KdV equation degenerates to a linear equation with a travelling wave at the critical point. The kink wave manifests itself in that traffic congestion is uniquely determined when the traffic flow is around the critical point.

Unlike the preceding studies on the OV model, in which the local density is induced by an external perturbation, the local density wave is formed by the density fluctuation of the traffic flow itself in our work.

2. Model and stability analysis

The OV model is quite a simple one whose equation exhibits traffic congestion. In the model, the acceleration of every car is determined by its velocity v_n and optimal speed $V(b_n)$, depending on the headway b_n as follows [2]:

$$\dot{v}_n = a[V(b_n) - v_n]. \tag{1}$$

In this equation the overdot represents differentiation with respect to time, $V(b_n) = \tanh(b_n - h_c) + \tanh(h_c)$ is the OV with safety distance h_c and a is the driver's sensitivity, which equals the inverse of the driver's reaction time [2].

It is generally believed that the headway b and the traffic flow density ρ are correlated with each other. It is noted that $\rho(x)$ is the associated position function from which we can find the vehicles. Following this, the conservation equation can be written as

$$\rho_t + (\rho v)_x = 0. \tag{2}$$

Here v is the speed of the traffic flow. The headway b can be written as a perturbation series [17]. Inserting the approximate expression about headway b into equation (1), we can obtain the traffic flow dynamics equation as follows:

$$v_t + v v_x = a[V(1/\rho) - v] - aV'(1/\rho)\left[\frac{\rho_x}{2\rho^3} + \frac{\rho_{xx}}{6\rho^4}\right]. \tag{3}$$

Writing the optimal speed as a function of density ρ, $\bar{V}(\rho) = V(1/\rho) = \tanh(1/\rho - h_c) + \tanh(h_c)$, equation (3) can be rewritten as [17]

$$v_t + v v_x = a[\bar{V}(\rho) - v] + a\bar{V}'(\rho)\left[\frac{\rho_x}{2\rho} + \frac{\rho_{xx}}{6\rho^2}\right]. \tag{4}$$

The above equation is a nonlinear one, and the ρ_{xx} term will cause a dispersive effect.

Let us assume that traffic flow is initially in a state differing infinitesimally from homogeneous flow. We decompose this flow into a linear combination of Fourier modes, each of which grows or decays with its own growth rate. Thus we write [11]

$$\begin{pmatrix} \rho(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} \rho_h \\ v_h \end{pmatrix} + \sum_k \begin{pmatrix} \hat{\rho}_k \\ \hat{v}_k \end{pmatrix} \exp(\mathrm{i}kx + \sigma_k t). \tag{5}$$

Substituting this expression into the governing equations (2) and (4) leads to the perturbation equations:

$$\hat{\rho}_k \sigma_k + \rho_k \hat{v}_k(\mathrm{i}k) + v_h \hat{\rho}_k(\mathrm{i}k) = 0, \tag{6}$$

$$\hat{v}_k \sigma_k + v_h(\mathrm{i}k)\hat{v}_k = a[\bar{V}'(\rho_h)\hat{\rho}_k - \hat{v}_k] + a\bar{V}'(\rho_h)\left[\frac{\mathrm{i}k\hat{\rho}_k}{2\rho_h} - \frac{k^2\hat{\rho}_k}{6\rho_h^2}\right]. \tag{7}$$

If the above equations in $\hat{\rho}_k$ and $\hat{v}_k$ are linearized, the coupled equations have no nontrivial solution unless their coefficient determinant is zero, i.e.

$$(\sigma_k + \mathrm{i}kv_h)^2 + a(\sigma_k + \mathrm{i}kv_h) + \mathrm{i}k\rho_h a\bar{V}'(\rho_h)\left(1 + \frac{\mathrm{i}k}{2\rho_h} - \frac{k^2}{6\rho_h^2}\right) = 0. \tag{8}$$

The traffic flow will remain stable as long as both roots of σ_k have negative imaginary parts. According to the Nyquist criterion, we obtain [11]

$$-2\rho_h^2\bar{V}'(\rho_h)\left(1 - \frac{k^2}{6\rho_h^2}\right)^2 < a. \tag{9}$$

At the same time, the stability condition of the traffic flow can be obtained as follows:

$$a = -2\rho_h^2\bar{V}'(\rho_h). \tag{10}$$

Using the rules for solving quadratic equations, we get the result of equation (8):

$$\sigma_k + \mathrm{i}kv_h = \frac{-a \pm \sqrt{a^2 - 4\mathrm{i}k\rho_h a\bar{V}'(\rho_h)\left(1 + \frac{\mathrm{i}k}{2\rho_h} - \frac{k^2}{6\rho_h^2}\right)}}{2}. \tag{11}$$

It is easy to draw the result that

$$\mathrm{Im}(\sigma_k) \approx -[v_h + \rho_h\bar{V}'(\rho_h)]k + \mathrm{O}(k^3). \tag{12}$$

From the imaginary part of σ_k, we see that the critical disturbance travels with a speed

$$c(\rho_h) = \bar{V}(\rho_h) + \rho_h\bar{V}'(\rho_h). \tag{13}$$

3. Nonlinear analysis

Now we consider the OV model with the long-wave expansion near the stable state determined by equation (10). The slow scales for space variable x and time variable t will be introduced in calculating the long-wave behaviour. For the case of $0 < \varepsilon \ll 1$, we can define the slow variables X and T as [11, 14]

$$X = \varepsilon(x - ct), \qquad T = \varepsilon^3 t. \tag{14}$$

Here $c(\rho_h) = \bar{V}(\rho_h) + \rho_h\bar{V}'(\rho_h)$. In the reference frame moving with a speed c, we set $\rho(x,t)$ and $v(x,t)$ as

$$\rho(x,t) = \rho_h + \varepsilon^2\hat{\rho}(X,T), \tag{15}$$

$$v(x,t) = v_h + \varepsilon^2\hat{v}(X,T). \tag{16}$$

Introducing these transforms into the fundamental equations (2) and (4), we obtain the following dynamical equation of the density fluctuation:

$$\hat{\rho}_T + [2\bar{V}' + \rho_h \bar{V}'']\hat{\rho}\hat{\rho}_X + \frac{\bar{V}'}{6\rho_h}\hat{\rho}_{XXX}$$
$$= \varepsilon \frac{\rho_h}{a}\left[\frac{\bar{V}'\alpha}{2\rho_h}\hat{\rho}_{XX} - \left[\rho_h \bar{V}'\bar{V}'' + \frac{\bar{V}'^2}{2} + \frac{a\bar{V}''}{4\rho_h}\right]\hat{\rho}^2_{XX} - \frac{\bar{V}'^2}{3\rho_h}\hat{\rho}_{4X}\right] \tag{17}$$

where $\bar{V}' = \bar{V}'(\rho_h) = -\rho_h^{-2}\mathrm{sech}^2\gamma$, $\bar{V}'' = \bar{V}''(\rho_h) = -2\bar{V}'\rho_h^{-1}(1 - \rho_h^{-1}\tanh\gamma)$ and $\alpha = -4\bar{V}'\tanh\gamma$ with $b_n - h_c = \gamma$. Neglecting the term with the factor ε in equation (17), we obtain the KdV equation as follows:

$$\hat{\rho}_T + [2\bar{V}' + \rho_h \bar{V}'']\hat{\rho}\hat{\rho}_X + \frac{\bar{V}'}{6\rho_h}\hat{\rho}_{XXX} = 0. \tag{18}$$

Using the real exponent approach [18], we obtain the one-soliton solution:

$$\hat{\rho} = A\mathrm{sech}^2[k(X + vT)], \tag{19}$$

where

$$A = \frac{12\frac{\bar{V}'(\rho_h)}{6\rho_h}}{2\bar{V}' + \rho_h\bar{V}''}k^2 = \frac{k^2}{\tanh\gamma}, \qquad v = -4\left(\frac{\bar{V}'}{6\rho_h}\right)k^2 = \frac{2}{3}\rho_h^{-3}\mathrm{sech}^2\gamma k^2.$$

When the distance is larger than the safety one, i.e. $\gamma > 0$, the solution is a soliton with the up wave crest because of $A > 0$. It indicates that the traffic flow will tend toward the critical point when the actual traffic density is a little larger than the safety distance. In contrast, we can derive $A < 0$ corresponding to $\gamma < 0$. It also indicates that the traffic flow will tend toward the critical point when the actual traffic density is a little smaller than that of the safety distance.

In order to discuss the effect of the order-ε correction, we write $T = Mt$, $X = By$ and $\hat{\rho} = Cu$. When

$$M = C = -6\rho_h^3 B^3/\mathrm{sech}^2\gamma, \qquad B = -\left(\frac{\mathrm{sech}^2\gamma}{12\rho_h^3\tanh\gamma}\right)^{1/5},$$

equation (17) will be transformed into

$$u_t + 6uu_y + u_{yyy} = \varepsilon\left[6B\rho_h^{-1}\tanh\gamma u_{yy} - \left(\rho_h^{-1}\tanh\gamma - \frac{1}{2}\right)\frac{18\rho_h^3B^4}{\mathrm{sech}^2\gamma}u_{2y}^2 + \frac{\rho_h^{-1}}{B}u_{4y}\right]. \tag{20}$$

Neglecting the term with ε, the above equation is simplified and its solution is $u = 2\eta^2\mathrm{sech}^2 Z$, where $Z = \eta(y - 4\eta^2 t)$.

By applying the direct approach, the dependence on time for the spectrum parameter η will be decided by the correction part, namely [19–22]

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\varepsilon}{4\eta}\int_{-\infty}^{+\infty}\mathrm{d}Z\,\mathrm{sech}^2 ZH, \tag{21}$$

where $H = 6B\rho_h^{-1}\tanh\gamma u_{yy} - (\rho_h^{-1}\tanh\gamma - \frac{1}{2})\frac{18\rho_h^3B^4}{\mathrm{sech}^2\gamma}u_{2y}^2 + \frac{\rho_h^{-1}}{B}u_{4y}$. After calculation in detail, the result becomes

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \varepsilon\eta^3\frac{16\rho_h^{-1}}{B}\left[-\frac{B^2\tanh\gamma}{5} - \left(\rho_h^{-1}\tanh\gamma - \frac{1}{2}\right)\frac{\rho_h^4B^5}{\mathrm{sech}^2\gamma}\frac{12}{7}\eta^2 + \left(-\frac{8}{105}\eta^2\right)\right]. \tag{22}$$

The amplitude η can be rewritten as $\eta = Bk$, and the above equation becomes

$$k_t = \varepsilon k^3\left(\frac{\mathrm{sech}^2\gamma}{12\rho_h^3\tanh\gamma}\right)^{3/5}16\rho_h^{-1}\left[\frac{\tanh\gamma}{5} - \left(\frac{1}{15} - \frac{\rho_h}{14\tanh\gamma}\right)k^2\right]. \tag{23}$$

Thus, the quality k^2 always increases until the headway reaches the safety distance.

When the distance is equal to the safety distance, equation (18) degenerates to a linear equation with the travelling wave solution:

$$\hat{\rho} = \hat{\rho}\left[-\left(\frac{|\bar{V}'|}{6\rho_h}\right)^{-1/3} X + T\right]. \tag{24}$$

The derivate V' of the OV has a maximal value at the turning point $b_n = h_c$. Because it is expected that the amplitude of the density wave scales as $A \propto \varepsilon$ near the critical point [13, 14, 23], we can reset $\rho(x,t)$ and $v(x,t)$ in the reference frame moving with speed c by writing

$$\rho(x,t) = \rho_0 + \varepsilon\hat{\rho}(X,T), \tag{25}$$

$$v(x,t) = v_0 + \varepsilon\hat{v}(X,T). \tag{26}$$

Here $c(\rho_0) = v_0 + \rho_0\bar{V}'(\rho_0)$ with the corresponding value at the critical point ρ_0 and v_0. Moreover, we have $a = 2(1-\varepsilon^2)$, $\bar{V}' = -\rho_0^{-2}$, $\bar{V}'' = 2\rho_0^{-3}$ and $\bar{V}''' = -6\rho_0^{-4} + 2\rho_0^{-6}$. Substituting these results into the governing equations (2) and (4), and neglecting the effect of the order-ε correction, we get the reduced equation as follows:

$$\hat{\rho}_T + \frac{\rho_0^{-5}}{3}\hat{\rho}_X^3 - \frac{\rho_0^{-4}}{6}\hat{\rho}_{XXX} = 0. \tag{27}$$

This is the MKdV equation of the density fluctuation around the critical point. According to different boundary conditions, the solution of the equation has the form of a kink solution [12–14, 23]:

$$\hat{\rho} = A^+ \tanh[\sqrt{1/2}A^+(X - A^{+2}T)], \tag{28}$$

or a soliton one [18, 19]:

$$\hat{\rho} = 2k\eta \operatorname{sech}[2k(X - 4k^2T)], \tag{29}$$

respectively. Thus, the car-following model with the OV describes that the density fluctuation in a traffic flow forms a local density wave automatically around the critical point.

4. Summary

The car-following model is one of the basic traffic models in which each driver can respond to the surrounding traffic conditions. In the model, the driving strategy of a driver depends on either the OV one or the desired one. Otherwise, Berg *et al* [17] have developed the OV model into a continuous one. Because the terms ρ_x and ρ_{xx} appear spontaneously in the nonlinear continuous model, the density fluctuation in traffic flow can be described by this model. It is certain that the density fluctuation will form the soliton or the kink wave under some conditions. Thus, the density wave can result in traffic congestion.

In this paper, two types of density waves induced by the traffic flow itself are found through investigating the OV model by linear and nonlinear analyses. One is the soliton wave and the other is the kink. The evolution of the soliton wave is also discussed. These results show that vehicles tend to reach the safety distance when they are away from it. Compared with other traffic models, the density fluctuation in our model is represented by explicit terms, which means that the congestion is an intrinsic property of traffic flow.

The model is applicable only for single-lane flow with no overtaking for single species. We can consider various ways of modifying our model for further studies. For example, the present model assumes that the sensitivity of drivers is identical. One could therefore adopt a model in which the sensitivity a depends on individual species for a multispecies system. It could be expected that more abundant phenomena will present themselves.

Acknowledgments

We are grateful to Professor N N Huang for his helpful discussions. This work was supported by the National Natural Science Foundation of China grant nos 19835040, 10075021, and the Plastic Shape Simulate & Mould Technology Laboratory.

References

[1] Gazis D C, Herman R and Rothery R W 1961 *Oper. Res.* **9** 545
[2] Bando M, Hasebe K, Nakayama A, Shibata A and Sugiyama Y 1995 *Phys. Rev.* E **51** 1035
[3] Helbing D and Tilch B 1998 *Phys. Rev.* E **58** 133
[4] Nagel K and Schreckenberg M 1992 *J. Physique* I **2** 2221
[5] Nagatani T 1995 *Phys. Rev.* E **51** 922
[6] Krauss S, Wagner P and Gawron C 1997 *Phys. Rev.* E **55** 5597
[7] Helbing D 1996 *Phys. Rev.* E **53** 2366
[8] Treiber M, Hennecke A and Helbing D 1999 *Phys. Rev.* E **59** 239
[9] Kerner B S and Konhäuser P 1993 *Phys. Rev.* E **48** R2335
[10] Kerner B S and Konhäuser P 1994 *Phys. Rev.* E **50** 54
[11] Kurtze D A and Hong D C 1995 *Phys. Rev.* E **52** 218
[12] Nagatani T 1998 *Physica* A **261** 599
[13] Komatsu T S and Sasa S I 1995 *Phys. Rev.* E **52** 5574
[14] Muramatsu M and Nagatani T 1999 *Phys. Rev.* E **60** 180
[15] Huang N N, Chen Z Y and Liu Z Z 1995 *Phys. Rev. Lett.* **75** 1395
[16] Dong G J and Liu Z Z 1996 *Opt. Commun.* **128** 8
[17] Berg P, Mason A and Woods A 2000 *Phys. Rev.* E **61** 1056
[18] Hereman W, Banerjee P P, Korpel A, Assanto G, Van Immerzeele A and Meerpoel A 1986 *J. Phys. A: Math. Gen.* **19** 607
[19] Huang N N 1996 *Theory of Solitons and Method of Perturbations* (Shanghai: Shanghai Scientific and Technological Education Publishing House)
[20] Herman R L 1990 *J. Phys. A: Math. Gen.* **23** 2327
[21] Yan J and Tang Y 1996 *Phys. Rev.* E **54** 6816
[22] Mann E 1997 *J. Math. Phys.* **38** 3772
[23] Nagatani T 2000 *Phys. Rev.* E **61** 3564
[24] Chowdhury D, Santen L and Schadschneider A 2000 *Phys. Rep.* **329** 199
[25] Helbing D 2001 *Rev. Mod. Phys.* **73** 1067